Valuation-Based Systems

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Outline

- Basics of Valuation-Based Systems
- Captain's Problem
- Valuations, Combination, and Marginalization in belief function theory
- Local Computation
- Removal and Conditionals
- Conditional Independence

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Basics of Valuation-based Systems

- A Valuation-based System (VBS) is a formal mathematical system for representation of and reasoning with knowledge.
- It has two parts. A static part that is concerned with representation of knowledge, and a dynamic part that is concerned with reasoning with knowledge
- The static part consists of:
  - Variables: A finite set Φ of variables $f X, Y, Z, \ldots g$. Subsets of Φ will be denoted by $r, s, t, \ldots$.
  - Valuations: A finite set Ψ of valuations $f \rho, \sigma, \tau, \ldots g$. Each valuation encodes knowledge about a subset of variables. Thus, we say, ρ is a valuation for r, where $r \quad \Phi$.
- A graphical representation of a VBS is called a valuation network

Basics of Valuation-based Systems

- The dynamic part consists of several operators:
  - Combination: $: \Psi \quad \Psi$ *!* Ψ that enables us to aggregate knowledge.
  - The combination operator has the following properties:
    - (Domain) If ρ is a valuation for *r*, and σ is a valuation for *s*, then $\rho \quad \sigma$ is a valuation for *r [s*
    - (Commutativity) $\rho \quad \sigma = \sigma \quad \rho$
    - (Associativity) $\rho \quad (\sigma \quad \tau) = (\rho \quad \sigma) \quad \tau$
  - The sequence in which knowledge is aggregated should make no difference.
  - The combination of all valuations, $\quad \Psi$, is called the joint valuation.

Basics of Valuation-based Systems

- Another operator is marginalization
- Marginalization: $-X : \Psi \rightarrow \Psi$ that allows us to coarsen knowledge marginalizing X out of the domain of a valuation.
- Properties of Marginalization
  - (Domain) If ρ is a valuation for r, and $X \in r$, then ρ^{-X} is a valuation for $r \setminus \{X\}$.
  - (Order does not matter) If ρ is a valuation for r, $X \in r$, and $Y \in r$, then $(\rho^{-X})^{-Y} = (\rho^{-Y})^{-X} = \rho^{-\{X,Y\}}$
  - (Local computation) If ρ and σ are valuations for r and s, respectively, $X \in r$, and $X \notin s$, then $(\rho \otimes \sigma)^{-X} = (\rho^{-X}) \otimes \sigma$
- We will sometimes denote $\rho^{-\{X,Y\}}$ by $\rho^{\downarrow r \setminus \{X,Y\}}$

Basics of Valuation-based Systems

- Making inference means finding marginals of the joint valuation for the variables of interest
- Thus, if X is a variable of interest, we compute $(\;\Psi)^{\downarrow X}$ by marginalizing all the other variables in $\Phi \setminus \{X\}$ out of the joint valuation $\;\Psi$.

Basics of Valuation-based Systems

- VBS is an abstraction of several uncertainty calculi
  - propositional calculus
  - probability theory
  - belief function theory
  - Spohn's epistemic belief calculus
  - possibility theory
- VBS can also be considered as an abstraction of
  - optimization using dynamic programming
  - Bayesian decision theory
  - Solving systems of equations
  - relational database theory

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Captain's Problem

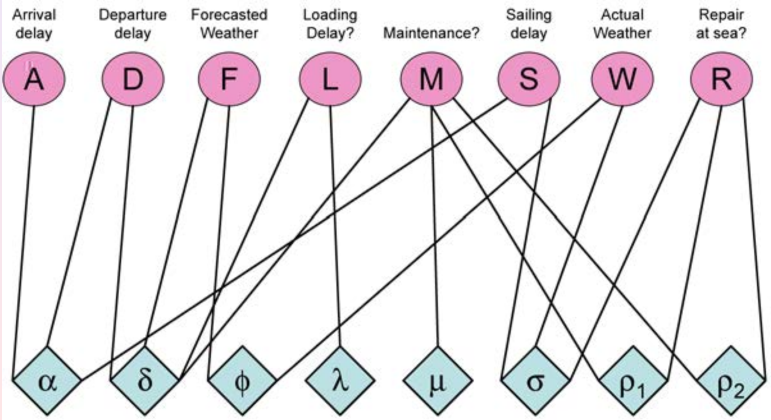
- Captain's Problem (R. Almond, Graphical Belief Modeling, Chapman and Hall, 1995)
  - A ship's captain is concerned about how many days his ship may be delayed before arrival at a destination.
  - The delay in arrival may be a result of delay in departure and/or delay in sailing.
  - Delay in departure may be a result of maintenance (at most 1 day), delay in loading (at most 1 day) or due to forecast of bad weather (at most I day).
  - Delay in sailing may be a result of bad weather (at most 1 day) and/or whether repairs may be needed at sea (at most 1 day).
  - If maintenance is done before sailing, chances of repairs at sea is less likely.
  - Weather forecast says small chance of bad weather (.2), good chance of good weather (0.6). Forecast is 80% reliable.
  - Captain has some knowledge of loading delay, and whether maintenance is done before departure.

Captain's Problem

- Variables
  - A (arrival delay), $\Omega_A = \{0, 1, 2, 3, 4, 5, 6\}$.
  - D (departure delay), $\Omega_D = \{0, 1, 2, 3\}$.
  - S (sailing delay), $\Omega_S = \{0, 1, 2, 3\}$.
  - L (is loading delayed?), $\Omega_L = \{t, f\}$.
  - F (weather forecast), $\Omega_F = \{b, g\}$.
  - W (actual weather), $\Omega_W = \{b, g\}$.
  - M (is maintenance done before sailing?), $\Omega_M = \{t, f\}$.
  - R (is repair at sea needed?), $\Omega_R = \{t, f\}$.

Captain's Problem

- Valuation Network: A bipartite graph with variables and valuations as nodes. Each valuation is linked to the variables in its domain.



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Belief Function Theory

- valuations in belief function theory
  - basic probability assignment (bpa) μ
  - belief function β
  - plausibility function π
  - commonality function χ
- Suppose $s \subseteq \Phi$. The state space of s is $\Omega_s = \times\{\Omega_X | X \in s\}$. Let 2^{Ω_s} denote the set of all subsets of Ω_s. A bpa μ for s is a function $\mu : 2^{\Omega_s} \to [0, 1]$ such that:
  - $\mu(\emptyset) = 0$,
  - $\mu(\mathbf{x}) \geq 0$, and
  - $\Sigma\{\mu(\mathbf{x}) | \mathbf{x} \in 2^{\Omega_s}\} = 1$.
- Subsets $\mathbf{x} \in 2^{\Omega_s}$ such that $\mu(\mathbf{x}) > 0$ are called focal sets of μ.

Belief Function Theory

- A function $\chi : 2^{\Omega_s} \to [0, 1]$ is a commonality function for s if there exists a bpa function μ for s such that

$$\chi(\mathbf{x}) = \Sigma\{\mu(\mathbf{y}) | \mathbf{y} \supseteq \mathbf{x}\}$$

- Suppose μ is a bpa for $\{W\}$ such that $\mu(\{g, b\}) = 1$. Then the corresponding commonality function χ for $\{W\}$ is $\chi(\emptyset) = 1$, $\chi(\{g\}) = 1$, $\chi(\{b\}) = 1$, $\chi(\{g, b\}) = 1$.

Captain's Problem

- Consider the piece of knowledge: Arrival delay is sum of departure delay and sailing delay
- We model this piece of knowledge by a bpa α for fA, D, Sg such that

$$\begin{aligned}\alpha(f(0,0,0),(1,1,0),(2,2,0),(3,3,0),\\ (1,0,1),(2,1,1),(3,2,1),(4,3,1),\\ (2,0,2),(3,1,2),(4,2,2),(5,3,2),\\ (3,0,3),(4,1,3),(5,2,3),(6,3,3)g) = 1.\end{aligned}$$

- α has one focal set. Such bpa are called deterministic.

Captain's Problem

- Loading delay, bad weather forecast, and maintenance each adds one day to departure delay
- We model this piece of knowledge by a bpa δ for $\{D, L, F, M\}$ such that

$$\delta(\{(0, f, g, f), (1, t, g, f), (1, f, b, f), (1, f, g, t), (2, f, b, t), (2, t, g, t), (2, t, g, f), (3, t, b, t)\}) = 1.$$

Captain's Problem

- At least 90% of the time, bad weather and repair at sea each adds 1 day to sailing delay
- We model this by bpa σ for $\{S, A, R\}$ such that

$$\begin{aligned} \sigma(\{(0, g, f), (1, b, f), (1, g, t), (2, b, t)\}) &= 0.9, \\ \sigma(\Omega_{\{S,A,R\}}) &= 0.1 \end{aligned}$$

Captain's Problem

- Forecast is 80% reliable
- This piece of knowledge is represented by bpa ϕ_1 for $\{F, W\}$ such that

$$\begin{aligned}\phi_1(\{(b, b), (g, g)\}) &= 0.8,\\ \phi_1(\Omega_{\{F,W\}}) &= 0.2.\end{aligned}$$

- Forecast predicts bad weather with chance 0.2 and good weather with chance 0.6
- This piece of knowledge is represented by bpa ϕ_2 for $\{F\}$ such that

$$\begin{aligned}\phi_2(\{b\}) &= 0.2,\\ \phi_2(\{g\}) &= 0.6,\\ \phi_2(\Omega_{\{F\}}) &= 0.2.\end{aligned}$$

Captain's Problem

- Loading is delayed with chance 0.3, and on schedule with chance 0.5.
- This piece is model by bpa λ for $\{L\}$ such that

$$\begin{aligned}\lambda(\{t\}) &= 0.3,\\ \lambda(\{f\}) &= 0.5,\\ \lambda(\Omega_{\{L\}}) &= 0.2.\end{aligned}$$

- No maintenance was done on the ship prior to departure
- This piece of knowledge is represented by bpa μ for $\{M\}$ such that

$$\mu(\{f\}) = 1.$$

Captain's Problem

- Consider valuation ρ_1 for fM, Rg as follows (If maintenance was done prior to sailing, then chances of repair at sea is between 10 and 30%)

$$
\begin{aligned}
\rho_1(f(t,t),(f,t),(f,f)g) &= 0.1,\\
\rho_1(f(t,f),(f,t),(f,f)g) &= 0.7,\\
\rho_1(\Omega_{\{M,R\}}) &= 0.2.
\end{aligned}
$$

- Consider valuation ρ_2 for fM, Rg as follows (If maintenance was not done prior to sailing, then chances of repair at sea is between 20 and 80%)

$$
\begin{aligned}
\rho_2(f(f,t),(t,t),(t,f)g) &= 0.2,\\
\rho_2(f(f,f),(t,t),(t,f)g) &= 0.2,\\
\rho_2(\Omega_{\{M,R\}}) &= 0.6.
\end{aligned}
$$

Belief Function Theory

- Combination in belief function theory is Dempster's rule of combination, also called product-intersection rule:
  - The product of the bpa values is assigned to the intersection of the subsets, The probability assigned to the ; is reduced to 0, and the remaining probabilities are normalized so they sum to 1.
- In terms of commonality functions, Dempster's rule of combination is pointwise multiplication followed by normalization.

Captain's Problem

- Suppose μ is a bpa for M that represents that maintenance was not done prior to sailing, i.e.,

$$\mu(\{f\}) = 1 \Rightarrow \mu^{\{M,R\}}(\{(f,t),(f,f)\}) = 1$$

- Now consider $\mu \quad \rho_1$

$\mu \quad \rho_1$	$\{(t,t),(f,t),(f,f)\}$ 0.1	$\{(t,f),(f,t),(f,f)\}$ 0.7	$\Omega_{\{M,R\}}$ 0.2
$\{(f,t),(f,f)\}$ 1	$\{(f,t),(f,f)\}$ 0.1	$\{(f,t),(f,f)\}$ 0.7	$\{(f,t),(f,f)\}$ 0.2

i.e.,$(\mu \quad \rho_1)(\{(f,t),(f,f)\} = 1$

Captain's Problem

- Now consider $\mu \quad \rho_2$

$\mu \quad \rho_2$	$f(f,t),(t,t),(t,f)g$ 0.2	$f(f,f),(t,t),(t,f)g$ 0.2	$\Omega_{\{M,R\}}$ 0.6
$f(f,t),(f,f)g$ 1	$f(f,t)g$ 0.2	$f(f,f)g$ 0.2	$f(f,t),(f,f)g$ 0.6

$$\begin{aligned} \text{i.e.,}(\mu \quad \rho_2)(f(f,t)g) &= 0.2, \\ (\mu \quad \rho_2)(f(f,f)g) &= 0.2, \\ (\mu \quad \rho_2)(f(f,t),(f,f)g) &= 0.6 \end{aligned}$$

Belief Function Theory

- Dempster's rule satisfies all properties of combination
  - (Domain) If μ_1 is a bpa for s_1 and μ_2 is a bpa for s_2, then $\mu_1 \quad \mu_2$ is a bpa for s_1 [s_2
  - (Associativity) $\mu_1 \quad (\mu_2 \quad \mu_3) = (\mu_1 \quad \mu_2) \quad \mu_3$
  - (Commutativity) $\mu_1 \quad \mu_2 = \mu_2 \quad \mu_1$

Belief Function Theory

- Marginalization in belief function theory is addition.
- Projection of States: If **a** $2\ \Omega_s$, and $X\ 2\ s$, then $\mathbf{a}^{\downarrow s \setminus \{X\}} = \mathbf{a}^{-X}$ is the state of $s\ n\ f X g$ obtained from **a** by dropping the state of X.
- If μ is a bpa for s, and $X\ 2\ s$, then μ^{-X} is defined as follows:

$$\mu^{-X}(\mathbf{a}) = \Sigma f \mu(\mathbf{y}) j \mathbf{y}^{-X} = \mathbf{a} g$$

Captain's Problem

- Consider ρ_1 for $\{M, R\}$

$$\begin{aligned}\rho_1(\{(t,t),(f,t),(f,f)\}) &= 0.1,\\ \rho_1(\{(t,f),(f,t),(f,f)\}) &= 0.7,\\ \rho_1(\Omega_{\{M,R\}}) &= 0.2.\end{aligned}$$

- Then ρ_1^{-M} for $\{R\}$ is as follows:

$$\rho^{-M}(\{t, f\}) = 1.$$

- And ρ_1^{-R} for $\{M\}$:

$$\rho^{-R}(\{t, f\}) = 1.$$

- Thus, ρ_1 by itself tells us nothing about M or R.

Captain's Problem

- Now consider $\mu \quad \rho_1$ for $f M, Rg$:

$$(\mu \quad \rho_1)(f(f, t), (f, f)g) = 1.$$

- Then, $(\mu \quad \rho_1)^{-M}$ for $f Rg$ is given by:

$$(\mu \quad \rho_1)(f t, f g) = 1.$$

i.e., $(\mu \quad \rho_1)^{-M}$ tells us nothing about R.

Captain's Problem

- Now consider $\mu \quad \rho_2$ for fM, Rg:

$$
\begin{aligned}
(\mu \quad \rho_2)(f(f, t)g) &= 0.2, \\
(\mu \quad \rho_2)(f(f, f)g) &= 0.2, \\
(\mu \quad \rho_2)(f(f, t), (f, f)g) &= 0.6.
\end{aligned}
$$

- Then, $(\mu \quad \rho_2)^{-M}$ for fRg is given by:

$$
\begin{aligned}
(\mu \quad \rho_2)^{-M}(ftg) &= 0.2, \\
(\mu \quad \rho_2)^{-M}(ffg) &= 0.2, \\
(\mu \quad \rho_2)^{-M}(ft, fg) &= 0.6.
\end{aligned}
$$

Belief Function Theory

- The definition of marginalization of bpa function satisfies the properties of marginalization:
  - (Domain) If ρ is a bpa for r, and $X \in r$, then ρ^{-X} is a bpa for $r \setminus \{X\}$.
  - (Order does not matter) If ρ is a bpa for r, $X \in r$, and $Y \in r$, then $(\rho^{-X})^{-Y} = (\rho^{-Y})^{-X} = \rho^{-\{X,Y\}} = \rho^{\downarrow r \setminus \{X,Y\}}$.
  - (Local computation) If ρ and σ are bpa's for r and s, respectively, $X \in r$, and $X \notin s$, then $(\rho \quad \sigma)^{-X} = (\rho^{-X}) \quad \sigma$.

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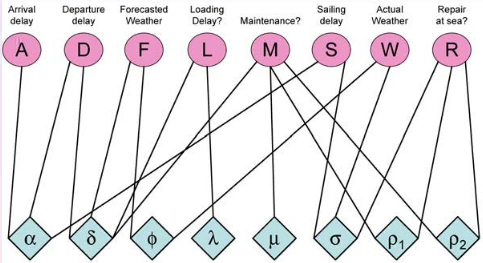
Local Computation

- Making inference means finding marginals of the joint valuation $\otimes\Psi$ for the variables of interest.
- If there are many variables in Φ, computing the joint valuation $\otimes\Psi$ for Φ is intractable.
- However, one can compute the marginal of the joint for X, $(\otimes\Psi)^{\downarrow X}$, without computing the joint explicitly, using so-called local computation.
- The axiom that allows local computation is the local computation axiom:
  If ρ and σ are bpa's for r and s, respectively, $X \in r$, and $X \notin s$, then $(\rho \otimes \sigma)^{-X} = (\rho^{-X}) \otimes \sigma$.

Local Computation

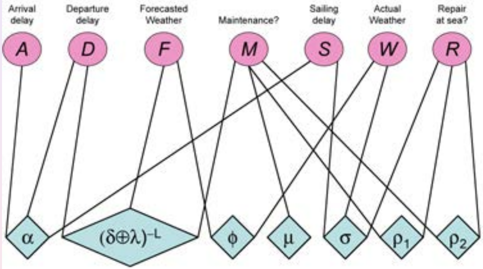
- Consider the Captain's problem. We would like to compute the marginal of the joint for *A*. So we have to marginalize all other variables from the joint.



- Consider *L*. It is in the domain of δ and λ only. The local computation axiom guarantees that if we replace δ and λ by $(\delta \quad \lambda)^{-L}$, then the product of all valuations will give us $(\quad \Psi)^{-L}$.

Local Computation

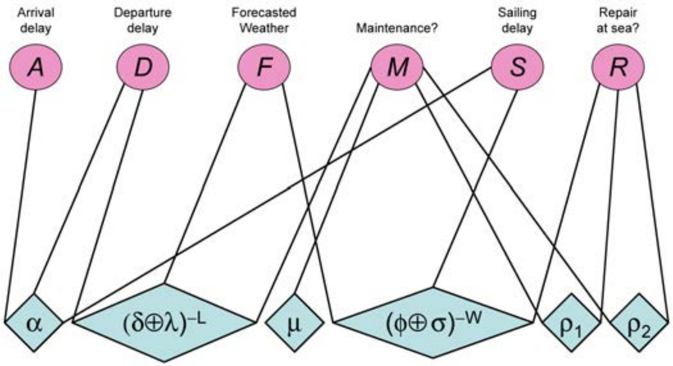
- The reduced VN is as follows:



- We can recursively remove all but *A* from the VN.
- Consider *W*. It is in the domain of ϕ for $\{F, W\}$ and σ for $\{S, W, R\}$. Thus, $(\phi \quad \sigma)^{-W}$ will be a bpa for $\{F, S, R\}$.

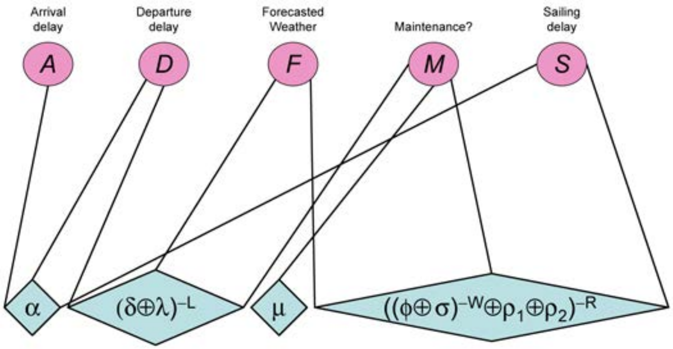
Local Computation

- After deletion of $\{L, W\}$:



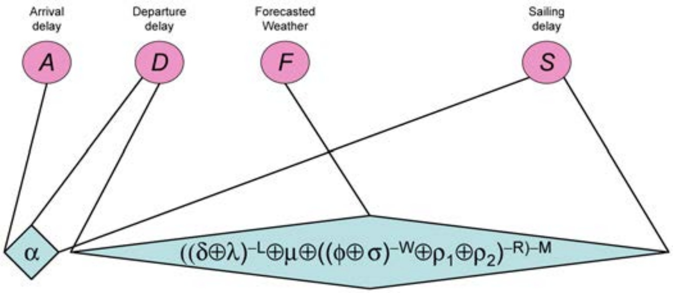
Local Computation

- After deletion of fL, W, Rg:



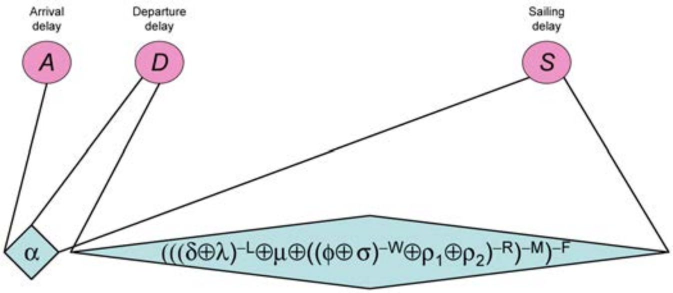
Local Computation

- After deletion of $f L, W, R, M g$:



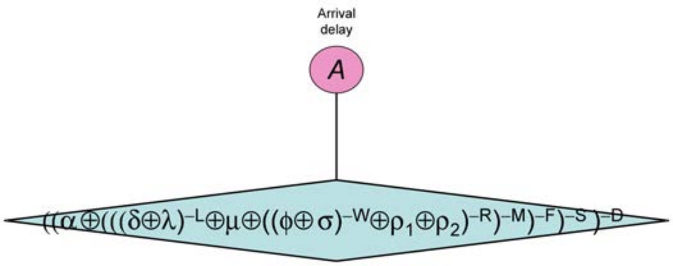
Local Computation

- After deletion of $f L$, W, R, M, Fg:



Local Computation

- After deletion of $f L, W, R, M, F, S, Dg$:



- If we combine all valuations, we have the marginal of the joint for A.

Local Computation

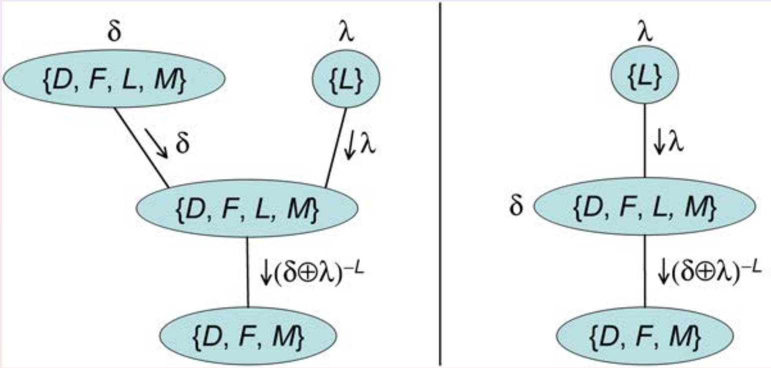
- In finding the marginal for A, we used deletion sequence $L\ W\ R\ M\ F\ S\ D$.
- The order does not matter axiom allows us to use any deletion sequence (and obtain the same marginal).
- Some deletion sequences involve less computation than others.
- Finding an optimal deletion sequence is a hard problem.
- So we use heuristics to select a sequence such as one-step-look-ahead: The variable to be marginalized next is the one that leads to combination on the smallest domain.

Local Computation

- If we can find the marginal for A, we can find the marginal for any other variable in a similar manner.
- However, there may be duplication of computations.
- We can avoid duplication by saving intermediate computations in a data structure called a “join tree.”
- A join tree is a tree with subsets of variables as nodes having the property that if a variable appears in two different nodes, then it appears in all nodes in the path between the two nodes.

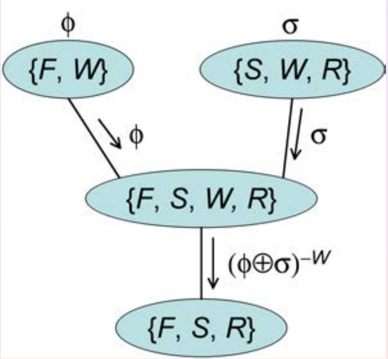
Local Computation

- Consider deletion of L. We can describe the computation as messages between nodes as follows:



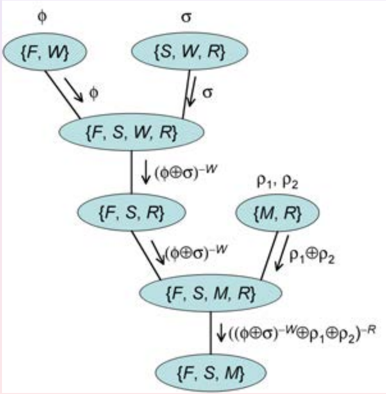
Local Computation

- Similarly, deletion of W can be described as follows:



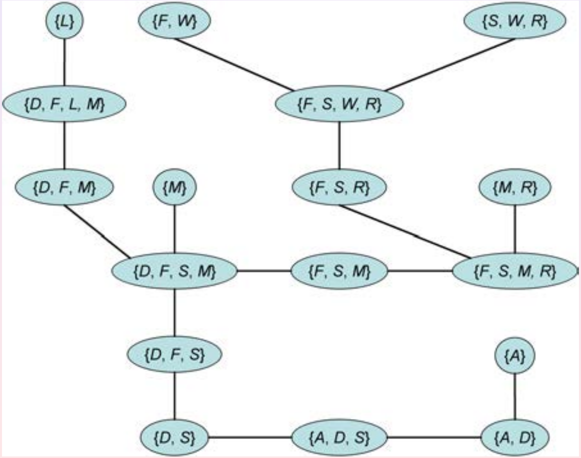
Local Computation

- Similarly, deletion of *R* can be described as follows:



Local Computation

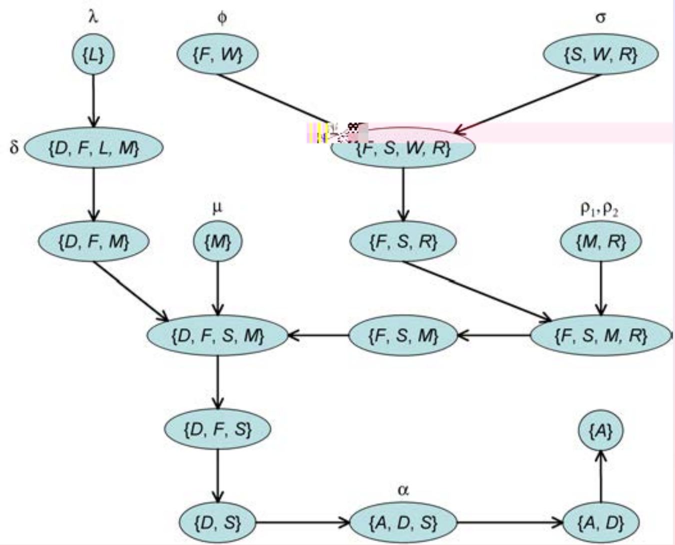
- The tree thus constructed is a join tree.



Local Computation

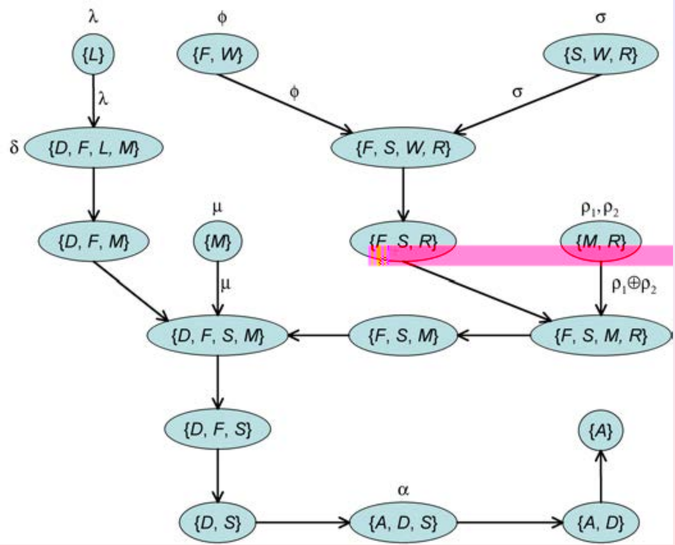
- To find the marginal for a variable, say A, orient all edges toward A, which is now the root of the tree.
- Each node sends a message to its inward neighbor, which is the combination of all messages it receives from its outward neighbors plus what it has suitably marginalized.
- Timing: Leaves (nodes with no outward neighbors) can send messages right away. Non-leaves have to wait till they receive a message from all its outward neighbors.
- Process is finished when the root has received a message from its outward neighbors. The root then combines all the messages it receives plus what it has.

- At the beginning:

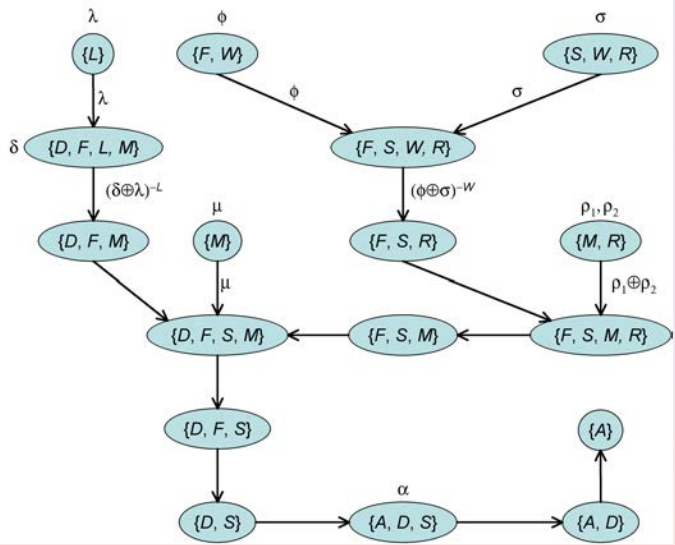


Local Computation

- Time step 1:



- Time step 2:


λ
ϕ
σ
{L}
{F, W}
{S, W, R}
λ
ϕ
σ
δ
{D, F, L, M}
{F, S, W, R}
(δ⊕λ)^{-L}
(ϕ⊕σ)^{-W}
μ
ρ1, ρ2
{D, F, M}
{M}
{F, S, R}
{M, R}
μ
ρ1⊕ρ2
{D, F, S, M}
{F, S, M}
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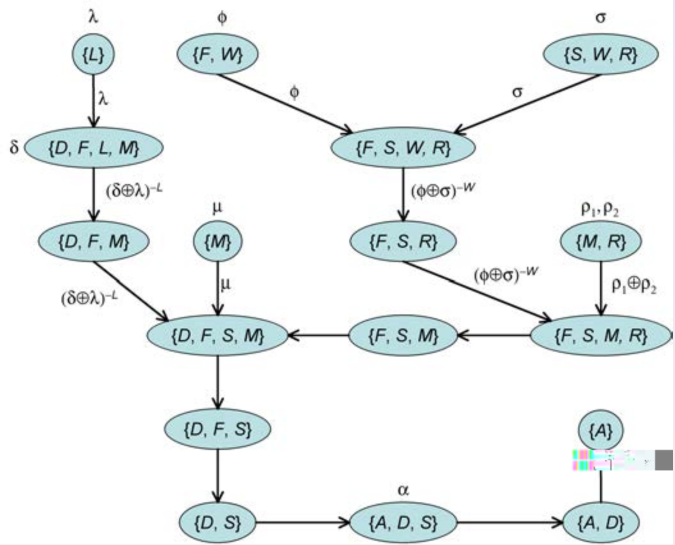
{A}
α
{D, S}
{A, D, S}
{A, D}

- Time step 3:


λ
{L}
λ
ϕ
{F, W}
ϕ
σ
{S, W, R}
σ
δ
{D, F, L, M}
{F, S, W, R}
(δ⊕λ)^{-L}
(ϕ⊕σ)^{-W}
μ
{D, F, M}
{M}
{F, S, R}
ρ1, ρ2
{M, R}
μ
(ϕ⊕σ)^{-W}
ρ1⊕ρ2
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{F, S, M}
{F, S, M, R}
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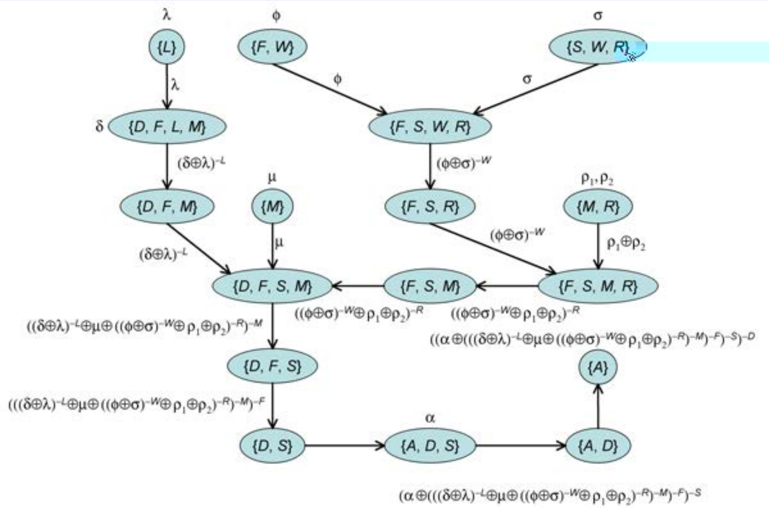
{A}
α
{D, S}
{A, D, S}
{A, D}

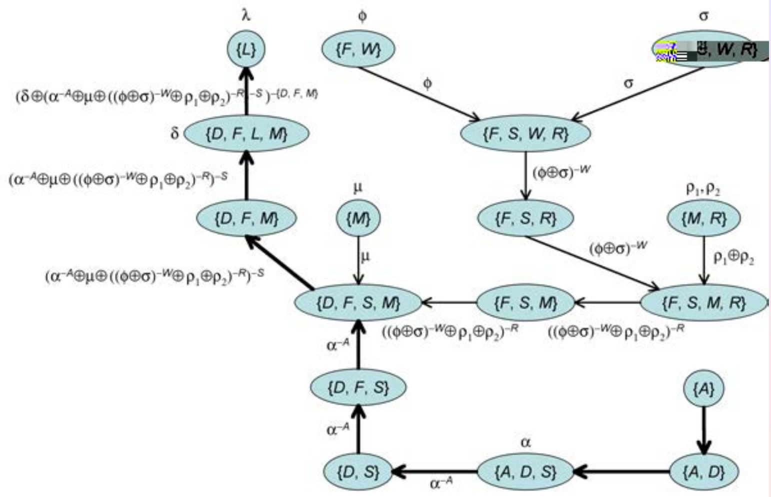
- End of propagation for *A*:



Local Computation

- Suppose we now wish to find the marginal for L.
- We now re-orient some of the edges so L is the root, and compute new messages for the re-oriented edges.

- End of propagation for *L*:



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Removal and Conditionals

- We now define a removal operator, which we will use to define conditionals.
- Removal $: \Psi \quad \Psi \; ! \; \Psi$ that allows us to remove one valuation from another
- The removal operator satisfies the following axioms:
  - (Domain): Suppose σ is a valuation for *s* and ρ is a valuation for *r*. Then $\sigma \quad \rho$ is a valuation for *r* [*s*.
  - (Identity): For each valuation ρ for *r*, there exists an identity valuation ι_ρ for *r* such that $\rho \quad \rho = \iota_\rho$
  - (Combination and Removal): $(\pi \quad \theta) \quad \rho = \pi \quad (\theta \quad \rho)$.
- We call $\sigma \quad \rho$, read as σ minus ρ, the valuation resulting after removing ρ from σ

Removal and Conditionals

- In belief function theory, removal is pointwise division of commonality functions followed by normalization.
- Suppose σ is a commonality function for s, and suppose ρ is a commonality function for r. Then, $\sigma \ominus \rho$ is a valuation for $r \cup s$ given by:

$$(\sigma \ominus \rho)(\mathbf{x}) = \begin{cases} K^{-1}\sigma(\mathbf{x}^{\downarrow s})/\rho(\mathbf{x}^{\downarrow r}) & \text{if } K > 0 \text{ and } \rho(\mathbf{x}^{\downarrow r}) > 0 \\ 0 & \text{if } K = 0 \text{ or } \rho(\mathbf{x}^{\downarrow r}) = 0 \end{cases}$$

where

$$K = \Sigma\{(-1)^{|\mathbf{x}|+1}\sigma(\mathbf{x}^{\downarrow s})/\rho(\mathbf{x}^{\downarrow r}) \mid \mathbf{x} \in 2^{\Omega_{r\cup s}} \text{ such that } \rho(\mathbf{x}^{\downarrow r}) > 0\}$$

Removal and Conditionals

- Suppose τ is a joint valuation for Φ, and suppose r and s are disjoint subsets of Φ. We will call $\tau^{\downarrow r \cup s} \quad \tau^{\downarrow r}$ the conditional for s given r with respect to τ, and denote it by $\tau(s\ j\ r)$
- If $r = \ ;$, let $\tau(s)$ denote $\tau(s\ j\ ;)$
- Properties of Conditionals
  - $\tau(s) = \tau^{\downarrow s}$
  - $\tau(r) \quad \tau(s\ j\ r) = \tau(r\ [\ s)$
  - $\tau(r\ j\ t) \quad \tau(s\ j\ r\ [\ t) = \tau(r\ [\ s\ j\ t)$
  - $\tau(s\ j\ r)^{\downarrow r} = \iota_{\tau(r)}$, i.e., $\tau(r) \quad (\tau(s\ j\ r)^{\downarrow r}) = \tau(r)$.
  - etc.

Outline

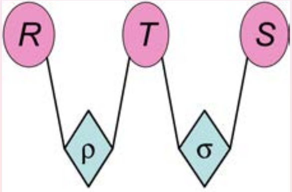
- Basics of Valuation-Based Systems
- Captain's Problem
- Valuations, Combination, and Marginalization in Belief Function Theory
- Local Computation
- Removal and Conditionals
- Conditional Independence

Conditional Independence

- DEFINITION: Suppose τ denotes the joint valuation, and suppose *r*, *s*, and *t* are disjoint subsets of variables. We say *r* is conditionally independent of *s* given *t* with respect to τ, written as $r\,?_{\tau} s \; j \; t$, if there exists ρ for $r \; [\; t$ and σ for $s \; [\; t$ such that

$$\tau(r \; [\; s \; [\; t) = \rho \quad \sigma$$

- For example, in the model below, $f R g \, ?_{\tau} \, f S g \; j \; f T g$, where $\tau = \rho \quad \sigma$

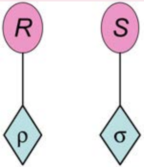


Conditional Independence

- A special case of conditional independence is independence when $t = ;$
- DEFINITON: Suppose τ denotes the joint valuation, and suppose r, and s are disjoint subsets of variables. We say r is independent of s with respect to τ, written as $r ?_\tau s$, if $r ?_\tau s \; j \; ;$, i.e., there exists ρ for r and σ for s such that

$$\tau(r \;[\; s) = \rho \quad \sigma$$

- For example, in the model below, $f R g ?_\tau f S g$, where $\tau = \rho \quad \sigma$

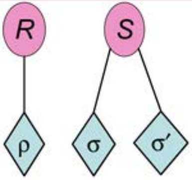


Conditional Independence

- What does it mean to say $fRg\,?_{T}\,fSg$?
- There are several semantics of independence
  - Irrelevance
  - Factorization
  - No double counting of evidence (when double counting matters)

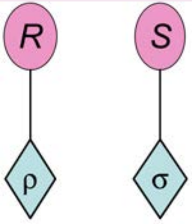
Conditional Independence

- Independence in terms of irrelevance
- In probability theory, $f R g ?_P f S g$ if $P(R \; j \; s_1) = P(R)$, for all $s_1 \; 2 \; \Omega_S$.
- Suppose $f R g ?_\tau f S g$. If we get some evidence about S, denoted say by σ' for $f S g$, then the marginal for R remains unchanged, i.e., if $\tau' = \rho \quad \sigma \quad \sigma'$, then $(\tau')(f R g) = (\rho \quad \sigma \quad \sigma')^{-S} = \rho \quad (\sigma \quad \sigma')^{-S} = \rho = \tau(f R g)$



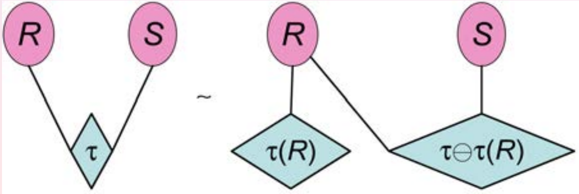
Conditional Independence

- Independence in terms of factorization
- In probability theory, $fRg ?_f fSg$ if $f(r, s) = g(r)h(s)$, where f is the joint PDF of R and S, r 2 Ω_R, and s 2 Ω_S.
- Suppose $fRg ?_\tau fSg$. Then $\tau = \rho \quad \sigma$ factors into a valuation ρ for fRg and a valuation σ for fSg.



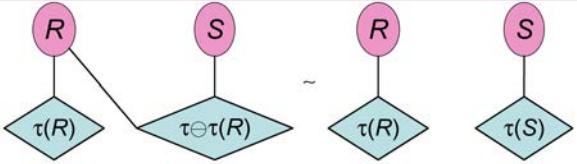
Conditional Independence

- Independence in terms of No double counting (when double counting matters).
- In general, $\sigma \oplus \sigma \neq \sigma$. So, we should be careful to not count a piece of evidence more than once.
- Suppose τ is a valuation for $\{R, S\}$. We can always factor τ as follows: $\tau(R)$ and $\tau(S \mid R) = (\tau \ominus \tau(R))$ and there is no double-counting.



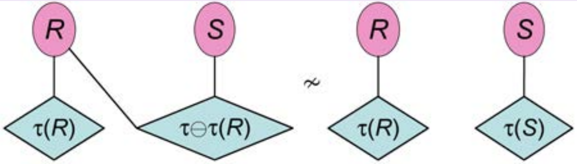
Conditional Independence

- If $f R g ?_{\tau} f S g$, then $\tau \quad \tau(R) = (\tau(R) \quad \tau(S)) \quad \tau(R) = \tau(S) \quad \iota_{\tau(R)}$. Thus, if we combine $\tau(R)$ and $\tau(S)$, we get τ, and there is no double counting.



Conditional Independence

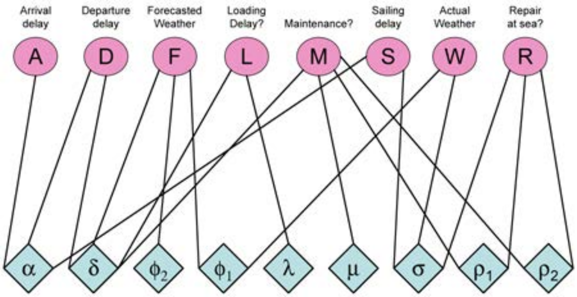
- If $f R g$ and $f S g$ are not independent, then $\tau(R) \quad \tau(S) \neq \tau$, since there is double-counting of evidence.



- $\tau = \tau(R) \quad (\tau \quad \tau(R))$. So $\tau(S) = (\tau(R) \quad (\tau \quad \tau(R)))^{\downarrow\{S\}}$. Thus, in combining $\tau(R)$ and $\tau(S)$, we are counting the information in $\tau(R)$ twice.
- Dempster's rule of combination should only be used to combine belief functions that are "distinct" or "independent", i.e., there is no double counting.

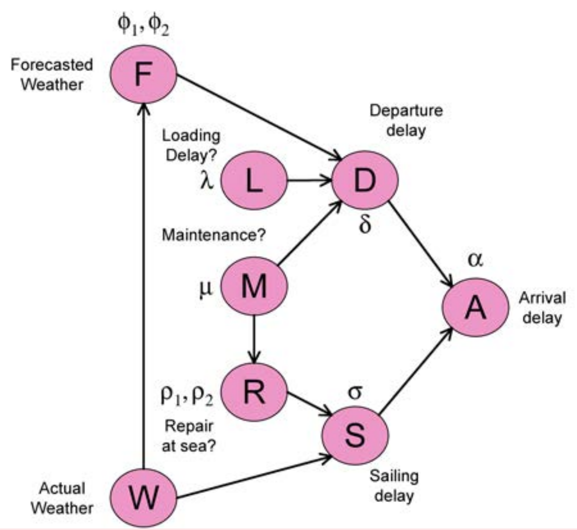
Conditional Independence

- One set of assumptions in the Captain's problem is that there is no double counting in the valuations in the model.



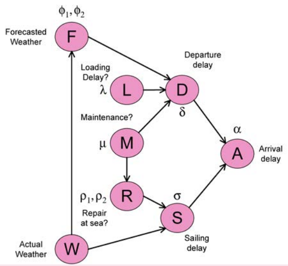
Conditional Independence

- The Captain problem can also be described by a causal DAG as follows:



Conditional Independence

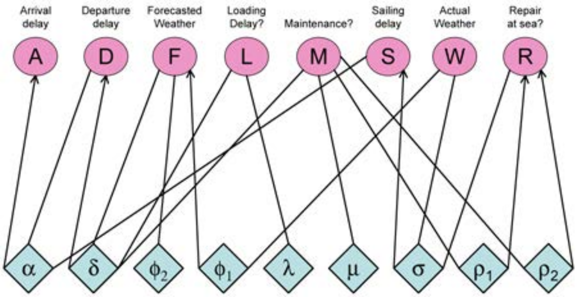
- We can deduce the following set of conditional independence assumptions from the causal model:



$$\{L\} \perp \{M\}, \quad \{W\} \perp \{L, M\}, \quad \{F\} \perp \{L, M\} \mid \{W\},$$
$$\{R\} \perp \{L, W, F\} \mid \{M\}, \quad \{D\} \perp \{W, S\} \mid \{F, L, M\},$$
$$\{S\} \perp \{L, M, F, D\} \mid \{R, W\}, \quad \{A\} \perp \{L, M, W, F, R\} \mid \{D, S\}$$

Conditional Independence

- We can derive the same set of conditional independencies from the VN if we include the fact that some of the valuations are conditionals (denoted by directed edges) as shown below.



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- Captain’s Problem: Almond, R, *Graphical Belief Modeling*, 1995, Chapman & Hall
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Questions

Any Questions?

Exercises

1. Explain why it is important to propagate in a join tree (see slide # 41) to get the correct marginals? Provide a simple example to demonstrate than propagation in a tree of subsets that is not a join tree will result in incorrect marginals
2. For the Captain's Problem, what is the marginal of the joint for *M*? Provide an answer symbolically similar to the marginal for *A* shown in slide #39.
3. Consider the case where we have valuations α for fAg, β for B, and a conditional valuation χ for C given A and B. Let τ denote the joint valuation for fA, B, Cg. Show that A and B are independent with respect to $\tau(fA, Bg) = \tau^{\downarrow\{A,B\}}$, but that this independence may be lost on observing $C = c$.