Exercises for the 3rd School on Belief functions
Random sets and belief functions

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September 27- October 2, 2015 – Stella Plage, France

1 Exercises

1. Consider $\Omega = \{1,2,3\}$ with the probability measure $P(\{1\}) = 0.3, P(\{2\}) = 0.5, P(\{3\}) = 0.2$ and the random set $\Gamma : \Omega \to \mathcal{P}(\{1,2,3\})$ given by
$$\Gamma(1) = \{1,2\}, \Gamma(2) = \{2,3\}, \Gamma(3) = \{1,2,3\}.$$
Determine the upper and lower probabilities of the sets $A = \{1\}, B = \{1,2\}$ and $C = \{2,3\}$.

2. Consider the belief function Bel on $\mathcal{P}(\{1,2,3\})$ given by:

A	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
Bel(A)	0.1	0.2	0	0.5	0.3	0.4	1

Determine a random set having Bel as its lower probability.

3. Consider $X = \{1,2,3,4\}$.
   (a) Let Π be the possibility distribution associated to the possibility distribution $\pi(1) = 0.3, \pi(2) = 0.5, \pi(3) = 1, \pi(4) = 0.7$. Determine its focal elements and its basic probability assignment.
   (b) Given the basic probability assignment $m(\{1\}) = 0.2, m(\{1,3\}) = 0.1, m(\{1,2,3\}) = 0.4, m(\{1,2,3,4\}) = 0.3$, determine the associated possibility measure and its possibility distribution.

4. Consider $\Omega = \{1,2,3\}$ with the uniform distribution, $\Gamma : \Omega \to \mathcal{P}(\{1,2\})$, $\Gamma(\omega) = \{1,2\}\ \forall \omega$.
   (a) Show that $P(\Gamma) = \{(0,1), (1/3,2/3), (2/3,1/3), (1,0)\}$.
   (b) Show that $M(P^*) = \{(\alpha, 1-\alpha) : \alpha \in [0,1]\}$.

   Let f be the Shannon entropy of a probability measure, $f(P) = -\sum_i P(x_i) log_2(P(x_i)$.
   (c) $f(P_{U_0}) \in f(P(\Gamma)) = \{0, f(1/3,2/3)\} = \{0, 0.92\}$.
   (d) $f(M(P_*)) = [0,1]$.

5. Consider $\Omega = \{1,2,3,4\} = X, \mathcal{A} = \mathcal{P}(\Omega), P$ the uniform distribution, and $\Gamma : \Omega \to \mathcal{P}(X)$ given by
$$\Gamma(1) = \{1\}, \Gamma(2) = \{1,3\}, \Gamma(3) = \{2,4\}, \Gamma(4) = \{3,4\}.$$
Determine the Choquet integral of the mapping f given by $f(1) = 10, f(2) = 4, f(3) = 0, f(4) = 2$ with respect to P^* and P_*, and give the measurable selections that attain this lower and upper integral.

6. Consider $\Omega = \{1,2,3\}$ with $P(\{1\}) = 0.2, P(\{2\}) = 0.3, P(\{3\}) = 0.5$ and the random set Γ given by $\Gamma(1) = \{1,2\}, \Gamma(2) = \{1,3\}, \Gamma(3) = \{2,3\}$.
   (a) Determine the lower and upper distribution functions of Γ.
   (b) Use them to find a probability measure Q such that $F_*(x) \leq F_Q(x) \leq F^*(x)\ \forall x \in \{1,2,3\}$ while $Q \notin M(P_*)$.

7. Consider the belief function associated with the basic probability assignment:

A	{1}	{2}	{1,2}	{1,3}	{2,3}	{1,2,3}
m(A)	1/9	1/9	1/9	1/6	1/6	1/3

Determine the extreme points of $M(Bel)$ and its Shapley value.

2 Solutions

1. $P_*(A) = 0, P^*(A) = 0.7, P_*(B) = 0.5, P^*(B) = 1, P_*(C) = 0.3, P^*(C) = 1.$

2. The basic probability assignment of Bel is given by

$$m(\{1\}) = 0.1 = m(\{1,2,3\}); m(\{2\}) = m(\{1,2\}) = m(\{1,3\}) = m(\{2,3\}) = 0.2.$$

   Thus, one random set having Bel as its lower probability would be $\Gamma : [0,1] \to \mathcal{P}(\{1,2,3\})$ given by:

$$\Gamma(\omega) = \begin{cases} \{1\} & \text{if } \omega \leq 0.1 \\ \{2\} & \text{if } 0.1 < \omega \leq 0.3 \\ \{1,2\} & \text{if } 0.3 < \omega \leq 0.5 \\ \{1,3\} & \text{if } 0.5 < \omega \leq 0.7 \\ \{2,3\} & \text{if } 0.7 < \omega \leq 0.9 \\ \{1,2,3\} & \text{if } \omega > 0.9, \end{cases}$$

   using Lebesgue measure and the Borel σ-field in the initial space.

3. (a) The basic probability assignment of Π is given by

$$m(\{3\}) = 0.3, m(\{3,4\}) = 0.2, m(\{2,3,4\}) = 0.2, m(\{1,2,3,4\}) = 0.3.$$

   (b) Π is determined by the possibility distribution $\pi(1) = 1, \pi(2) = 0.7, \pi(3) = 0.8, \pi(4) = 0.3$.

4. (a) The measurable selections of Γ are given by (using the notation $(U(1), U(2), U(3))$:

$$S(\Gamma) := \{(1,1,1), (1,1,2), (1,2,1), (2,1,1), (1,2,2), (2,1,2), (2,2,1), (2,2,2)\},$$

   whence $P(\Gamma) = \{(1,0), (\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}), (0,1)\}$.

   (b) Since $M(P^*)$ is the convex hull of $P(\Gamma)$, we deduce that $M(P^*) = \{(\alpha, 1-\alpha) : \alpha \in [0,1]\}$.

   (c) The entropy of $(1,0)$ and $(0,1)$ is 0, while the entropy of $(\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3})$ is 0.92.

   (d) The distribution $(0.5, 0.5) \in M(P^*)$ has entropy 1; then use that $M(P^*)$ is convex and f is continuous.

5. $(C) \int f dP_* = 3, (C) \int f dP^* = 6.5$. We attain these integrals with the measurable selections $U_1 = (1,3,4,3)$ and $U_2 = (1,1,2,4)$, using again the vector notation.

6. (a) The lower and upper distribution functions are:

	1	2	3
F_*	0	0.2	1
F^*	0.5	1	1

   (b) The distribution function F given by $F(1) = 0, F(2) = F(3) = 1$ belongs to the p-box (F_*, F^*). However, its associated probability measure is given by $Q(\{2\}) = 1 > P^*(\{2\}) = 0.8$, and as a consequence it does not belong to the core $M(P_*)$.

7. The extreme points are determined by the permutations:

Permutation σ	P_σ
$(1,2,3)$	$(1/9, 2/9, 2/3)$
$(1,3,2)$	$(1/9, 13/18, 1/6)$
$(2,1,3)$	$(2/9, 1/9, 2/3)$
$(2,3,1)$	$(13/18, 1/9, 1/6)$
$(3,1,2)$	$(5/18, 13/18, 0)$
$(3,2,1)$	$(13/18, 5/18, 0)$

   Using the formula of the Shapley value (or also making the average of the above extreme points), we obtain that the Shapley value is given by the distribution $Q(\{1\}) = Q(\{2\}) = \frac{13}{36}, Q(\{3\}) = \frac{5}{18}$.